MATH 2050C Mathematical Analysis I
2018-19 Term 2

Suggested Solution to Midterm

1(a)

The completeness property of $\mathbb{R}$ says that any non-empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.

1(b)

Since S is a non-empty subset of $\mathbb{R}$ which is bounded above, by the completeness property of $\mathbb{R}$, the supremum of S exists in $\mathbb{R}$. Let $u = \sup S$.

We first show that $-u$ is a lower bound for $-S$, hence $-S$ is bounded below. Since u is the supremum of S, in particular, it is an upper bound of S, which means that $s \leq u$ for all $s \in S$. Multiplying by -1 gives $-s \geq -u$ for all $s \in S$. Since every element of $-S$ has the form $-s$ for some $s \in S$, $-u$ is a lower bound of $-S$.

Next, we show that $-u$ is the greatest lower bound for $-S$, i.e. $-u = \inf(-S)$. In other words, we want to show that $-u + \epsilon$ is not a lower bound of $-S$ for any $\epsilon > 0$. Since u is the least upper bound of S, for any $\epsilon > 0$, $u - \epsilon$ is not an upper bound of S. Therefore, there exists some $s_0 \in S$, depending on ϵ, such that $u - \epsilon < s_0$. Multiplying by -1 gives $-u + \epsilon > -s_0$. Since $-s_0 \in -S$, $-u + \epsilon$ cannot be a lower bound for $-S$. Since $\epsilon > 0$ is arbitrary, this proves that $-u = \inf(-S)$.

2(a)

First of all, observe that $b^{\frac{1}{n}} > 1$ for all $n \in \mathbb{N}$ since $b > 1$. Therefore, we can write $b^{\frac{1}{n}} = 1 + d_n$ for some $d_n > 0$. By Bernouli's inequality, since $d_n > 0 > -1$,

$$b = (1 + d_n)^n \geq 1 + nd_n.$$

This implies that $d_n \leq \frac{b-1}{n}$ for each $n \in \mathbb{N}$.

To prove that $\lim(b^{\frac{1}{n}}) = 1$. Let $\epsilon > 0$ be fixed but arbitrary. By Archimedean Property, we can choose $K \in \mathbb{R}$ such that $K > \frac{b-1}{\epsilon} > 0$. For any $n \geq K$, we have

$$|b^{\frac{1}{n}} - 1| = |d_n| = d_n \leq \frac{b-1}{n} \leq \frac{b-1}{K} < \epsilon.$$

2(b)

Since $0 < a < 1$, we have $0 < a^n < 1$ for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$,

$$1 \leq (1 + a^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}}.$$

By (a), $\lim(2^{\frac{1}{n}}) = 1$. Also, we have $\lim(1) = 1$. By squeeze theorem, we have $\lim((1 + a^n)^{\frac{1}{n}}) = 1$ as well.

3(a)

We will prove the statement by mathematical induction. Note first that for $n = 1$, $1 \leq x_1 = 1 \leq 3$. Assume $1 \leq x_k \leq 3$ for some $k \in \mathbb{N}$. Then,

$$x_{k+1} = \frac{3 + 2x_k}{3 + x_k} = 1 + \frac{x_k}{3 + x_k} \geq 1$$

where the last inequality holds since $x_k \geq 1 \geq 0$. On the other hand,

$$x_{k+1} = \frac{3 + 2x_k}{3 + x_k} = 2 - \frac{3}{3 + x_k} \leq 2 \leq 3$$

where the second to the last inequality holds since $x_k \geq 1 \geq 0$.

3(b)

We will show that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ again by induction. Note that

$$x_1 = 1 \leq \frac{5}{4} = x_2.$$

So the statement is true for $n = 1$. Suppose $x_k \leq x_{k+1}$ for some $k \in \mathbb{N}$. Note that

$$x_{k+2} - x_{k+1} = \frac{3 + 2x_{k+1}}{3 + x_{k+1}} - \frac{3 + 2x_k}{3 + x_k} = \frac{3(x_{k+1} - x_k)}{(3 + x_{k+1})(3 + x_k)} \geq 0$$

where we have used the induction hypothesis that $x_k \leq x_{k+1}$ and that $x_n \geq 0$ for all $n \in \mathbb{N}$ by (a). By mathematical induction, we have $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

Combining with (a), (x_n) is an increasing sequence which is bounded above by 3. By Monotone Convergence Theorem, (x_n) converges to a unique limit $x \in \mathbb{R}$. Since $\lim(x_{n+1}) = \lim(x_n) = x$, by taking limit in the recursive relation. We obtain

$$x = \frac{3 + 2x}{3 + x}.$$

Rearranging gives the quadratic equation $x^2 + x - 3 = 0$, which yields

$$x = \frac{-1 + \sqrt{13}}{2} \quad \text{or} \quad x = \frac{-1 - \sqrt{13}}{2}.$$

The second solution is discarded as we know by (a) that $1 \leq x = \lim(x_n) \leq 3$. Therefore, $\lim(x_n) = \frac{-1+\sqrt{13}}{2}$.

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We prove by contradiction. Suppose $\sqrt{12}$ is not irrational. Then, there exists $m, n \in \mathbb{Z}$, $n \neq 0$, such that m and n are relatively prime and $\sqrt{12} = m/n$. Squaring both sides and rearranging gives

$$12n^2 = m^2.$$

Since 3 divides 12, it also divides m^2, and hence m as well because 3 is prime. Write $m = 3k$ for some $k \in \mathbb{Z}$. We have

$$12n^2 = m^2 = (3k)^2 = 9k^2.$$

Thus, $4n^2 = 3k^2$. Therefore, by the same argument, 3 divides n^2 and hence n. As a result, m and n are both divisible by 3 which contradicts the fact that they are relatively prime.

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We proceed by contradiction. Suppose on the contrary that (x_n) does not converge to x. Then, there exists $\epsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that

$$|x_{n_k} - x| \geq \epsilon_0 \qquad \text{for all } k \in \mathbb{N}. \tag{1}$$

On the other hand, since (x_{n_k}) is a subsequence of (x_n), by assumption there exists another subsequence $(x_{n_{k_\ell}})$ of (x_{n_k}) such that $\lim(x_{n_{k_\ell}}) = x$ as $\ell \to \infty$. This implies that for the particular $\epsilon_0 > 0$ above, we can find some $L \in \mathbb{N}$ such that

$$|x_{n_{k_\ell}} - x| < \epsilon_0 \qquad \text{for all } \ell \geq L.$$

However, since $(x_{n_{k_\ell}})$ is a subsequence of (x_{n_k}), every term in the sequence $(x_{n_{k_\ell}})$ also satisfies (1). Therefore, we obtain when $\ell = L$,

$$\epsilon_0 \leq |x_{n_{k_L}} - x| < \epsilon_0,$$

which is a contradiction.

6(a)

Since (x_n) is a bounded sequence, there exists some $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Therefore,

$$-M \leq s_m := \inf\{x_n : n \geq m\} \leq x_m \leq M$$

for all $m \in \mathbb{N}$. Thus, the sequence (s_m) is bounded above by M.

Next we show that (s_m) is an increasing sequence. Recall that $\inf S_1 \geq \inf S_2$ for any bounded subset $S_1 \subseteq S_2$ of $\mathbb{R}$. As

$$\{x_n : n \geq m + 1\} \subseteq \{x_n : n \geq m\}.$$

Taking infimum gives $s_{m+1} \geq s_m$ for any $m \in \mathbb{N}$. Therefore, (s_m) is an increasing sequence.

Since (s_m) is an increasing sequence which is bounded above, by Monotone Convergence Theorem (s_m) is convergent with

$$\lim(s_m) = \sup\{s_m : m \in \mathbb{N}\}.$$

6(b)

Let $x \in [0, 1]$ be fixed but arbitrary. By density of $\mathbb{Q}$, there exists some rational number $0 \leq q \leq 1$ in the interval $(x - \frac{1}{2}, x + \frac{1}{2})$ such that $q \neq x$. Define q_{n_1} to be this rational number.

To define the next term q_{n_2} in the subsequence, notice that there are only finitely many terms $q_1, q_2, \cdots, q_{n_1 - 1}$ before q_{n_1} in the sequence (q_n) and that any open interval contains infinitely many rational numbers (for example, by density of $\mathbb{Q}$). Therefore, there exists q_{n_2} with $n_2 > n_1$ such that q_{n_2} lies in the open interval between q_{n_1} and x such that $|q_{n_2} - x| < 1/3$.

Inductively, after q_{n_k} is fixed, we can choose $q_{n_{k+1}}$ such that $n_{k+1} > n_k$ and $q_{n_{k+1}} \neq x$ lies in the open interval between q_{n_k} and x such that $|q_{n_k} - x| \leq \frac{1}{k+1}$. As $\lim(\frac{1}{k+1}) = 0$, we have obtained a subsequence (q_{n_k}) of (q_n) such that $\lim(q_{n_k}) = x$.